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The Entropy of the Complex Scalar Field in a Charged Kerr Black Hole

Min-Ho Lee[1] and Jae Kwan Kim

Department of Physics, Korea Advanced Institute of Science and Technology

373-1 Kusung-dong, Yusung-ku, Taejon 305-701, Korea.

Abstract

By using the brick wall method we calculate the thermodynamic potential of the complex scalar field in a charged Kerr black hole. Using it we show that in the Hartle-Hawking state the leading term of the entropy is proportional to $\frac{A_H}{\epsilon^2}$, which becomes divergent as the system approaches the black hole horizon. The origin of the divergence is that the density of states diverges at the horizon.

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[1]e-mail : mhlee@chep6.kaist.ac.kr

By comparing the black hole physics with the thermodynamics Bekenstein showed that the black hole entropy is proportional to the horizon area [1, 2], and Hawking's discovery of the black hole evaporation confirmed that. In Euclidean path integral approach it was shown that the tree level contribution of the gravitation action gives the black hole entropy [3]. However the exact statistical origin of the Bekenstein-Hawking black hole entropy is unclear.

Recently many efforts have been concentrated on understanding the statistical origin of black hole thermodynamics, specially the black hole entropy by various methods [4]: the brick wall method, the conical singularity method, and the entanglement entropy method [5, 6]. The leading term of the entropy obtained by those methods is proportional to the surface area of the horizon. However the proportional coefficient diverges as the cut-off goes to zero. The divergence is because of an infinite number of states near the horizon, which can be explained by the equivalence principle [7].

For the black hole with a rotation the entropy of the neutral scalar field was calculated by authors [9]. It was shown that the leading term of the entropy, if the quantum field is at the Hartle-Hawking state, is proportional to the horizon area. In this paper to understand more deeply the black hole entropy we will investigate the entropy of the complex scalar field interacting with the gauge field A_μ in the charged Kerr black hole background [10].

Let us consider a minimally coupled complex scalar field with mass μ in thermal equilibrium at temperature $1/\beta$ in the charged Kerr black hole spacetime. The line element of the charged Kerr black hole spacetime and the electromagnetic vector potential in Boyer-Lindquist coordinates are given by [10, 11]

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a\sin^2\theta\ (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi$$

$$
\begin{aligned}
& + \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right] \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\ d\theta^2 \\
\equiv\ & g_{tt}(r,\theta)dt^2 + 2g_{t\phi}(r,\theta)dtd\phi + g_{\phi\phi}(r,\theta)d\phi^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2, \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
A_\mu \ =\ & -\frac{er}{\Sigma}\left[(dt)_\mu - a\sin^2\theta(d\phi)_\mu\right] \\
\equiv\ & A_t(r,\theta)(dt)_\mu + A_\phi(r,\theta)(d\phi)_\mu, \qquad (2)
\end{aligned}
$$

where

$$
\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 + e^2 - 2Mr, \tag{3}
$$

and e, a, and M are charge, angular momentum per unit mass, and mass of the spacetime respectively. The charged Kerr black hole spacetime has two Killing vector fields: the time-like Killing vector $\xi^\mu = (\partial_t)^\mu$ and the axial Killing vector $\psi^\mu = (\partial_\phi)^\mu$. We assume that $e^2 + a^2 < M^2$. In that case the charged Kerr black hole has the event horizon at $r = r_H = M + \sqrt{M^2 - a^2 - e^2}$ and the stationary limit surface at $r = r_0 = M + \sqrt{M^2 - e^2 - a^2\cos^2\theta}$.

The equation of motion of the field with mass μ is

$$
\left[(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu) - \mu^2\right]\Psi = 0. \tag{4}
$$

We assume that the scalar field is rotating with a constant azimuthal angular velocity Ω_0 and has a chemical potential Φ_0. The associated conserved quantities are the angular momentum J and the electric charge Q. For such a equilibrium ensemble of the states of the field the grand partition function is given by

$$
Z = \mathrm{Tr} e^{-\beta(H - \Omega_0 \cdot J - \Phi_0 Q)}. \tag{5}
$$

The thermodynamic potential of the system for particles with charge q (for anti-particle $q \to -q$) is given by

$$
W = \frac{1}{\beta}\sum_{j,m} d_{j,m} \ln\left(1 - e^{-\beta(\mathcal{E}_{j,m} - m\Omega_0 - q\Phi_0)}\right) \tag{6}
$$

or

$$W = \frac{1}{\beta}\sum_m \int_0^\infty d\mathcal{E} g(\mathcal{E}, m) \ln\left(1 - e^{-\beta(\mathcal{E} - m\Omega_0 - q\Phi_0)}\right), \tag{7}$$

where $g(\mathcal{E}, m)$ is the density of states for a given $\mathcal{E}$ and m.

To evaluate the thermodynamic potential we will follow the brick wall method of 't Hooft [5]. Following the brick wall method we impose a small cut-off h such that

$$\Psi(x) = 0 \quad \text{for} \quad r \le r_H + h. \tag{8}$$

To remove the infra-red divergence we also introduce another cut-off $L \gg r_H$ such that

$$\Psi(x) = 0 \quad \text{for} \quad r \ge L. \tag{9}$$

In the WKB approximation with $\Psi = e^{-i\mathcal{E}t + im\phi + iS(r,\theta)}$ the equation (4) in Lorentz gauge $\nabla_\mu A^\mu = 0$ yields the constraint [12]

$$p_r^2 = \frac{1}{g^{rr}}\left[-g^{tt}(\mathcal{E} + qA_t)^2 + 2g^{t\phi}(\mathcal{E} + qA_t)(m - qA_\phi) - g^{\phi\phi}(m - qA_\phi)^2 - g^{\theta\theta}p_\theta^2 - \mu^2\right], \tag{10}$$

where $p_r = \partial_r S$ and $p_\theta = \partial_\theta S$. The number of states for a given $\mathcal{E}$ is determined by p_θ, p_r and m. Therefore the thermodynamic potential (as in flat spacetime) is written as

$$\beta W = \int dr d\theta d\phi \sum_m \int \frac{dp_r dp_\theta}{(2\pi)^2} \ln\left(1 - e^{-\beta(\mathcal{E} - m\Omega_0 - q\Phi_0)}\right), \tag{11}$$

where

$$\mathcal{E} = \frac{1}{-g^{tt}}\left\{-g^{t\phi}p_\phi + \left[\left(g^{t\phi}p_\phi\right)^2 + (-g^{tt})\left(\mu^2 + g^{\phi\phi}p_\phi^2 + g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2\right)\right]^{1/2}\right\} - qA_t. \tag{12}$$

Here $p_\phi = m - qA_\phi$. Let us introduce new variables:

$$\bar{p}_\phi = \left(\frac{-\mathcal{D}}{g_{\phi\phi}^2}\right)^{1/2} p_\phi, \quad \bar{p}_r = \left(\frac{-\mathcal{D}}{g_{\phi\phi}g_{rr}}\right)^{1/2} p_r, \quad \bar{p}_\theta = \left(\frac{-\mathcal{D}}{g_{\phi\phi}g_{\theta\theta}}\right)^{1/2} p_\theta, \tag{13}$$

where $\mathcal{D} = g_{tt}g_{\phi\phi} - g_{t\phi}^2$. Then the thermodynamic potential, with the assumption that m is a continuous variable, is written as

$$
\begin{aligned}
&\beta W \\
= &\int dr d\theta d\phi \int \frac{d\bar{p}_r d\bar{p}_\theta d\bar{p}_\phi}{(2\pi)^3} \frac{g_{\phi\phi}^2 \sqrt{g_{\theta\theta} g_{rr}}}{(-\mathcal{D})^{3/2}} \ln \left(1 - e^{-\beta(\alpha \bar{p}_\phi + \sqrt{\bar{p}_\phi^2 + \bar{p}_r^2 + \bar{p}_\theta^2 + \bar{\mu}^2} - q[A_t + \Omega_0 A_\phi] - q\Phi_0)}\right) \\
= &\int dr d\theta d\phi \int \frac{d\bar{p}\bar{p}^2 d\bar{\theta} \sin\bar{\theta} d\bar{\phi}}{(2\pi)^3} \frac{g_{\phi\phi}^2 \sqrt{g_{\theta\theta} g_{rr}}}{(-\mathcal{D})^{3/2}} \ln \left(1 - e^{-\beta(\alpha \bar{p}\cos\bar{\theta} + \sqrt{\bar{p}^2 + \bar{\mu}^2} - q\Gamma)}\right) \qquad (14)
\end{aligned}
$$

in spherical coordinates in momentum space, where

$$
\alpha = \left(-\frac{g_{t\phi}}{g_{\phi\phi}} - \Omega_0\right)\left(\frac{g_{\phi\phi}^2}{-\mathcal{D}}\right)^{1/2}, \qquad \bar{\mu}^2 = \left(\frac{-\mathcal{D}}{g_{\phi\phi}}\right)\mu^2, \qquad \Gamma = \Phi_0 + A_t + \Omega_0 A_\phi. \qquad (15)
$$

Note that we must restrict the system to be in the region such that $1 \pm \alpha > 0$ or equivalently $g_{tt}^{'} \equiv g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0$. In the region such that $-g_{tt}^{'} > 0$ (called region I) the integral is convergent. But in the region such that $-g_{tt}^{'} \leq 0$ (called region II) the integral is divergent. These facts become more apparent if we investigate the momentum phase space. In the region I the possible points of p_i satisfying $\mathcal{E} + qA_t - \Omega_0 p_\phi = E$ for a given E are located on the following surface

$$
\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} + \frac{-g'_{tt}}{-\mathcal{D}}\left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}} E\right)^2 = \left(\frac{E^2}{-g'_{tt}} - \mu^2\right), \qquad (16)
$$

which is a ellipsoid, *a compact surface*. So the density of states $g(E)$ for a given E is finite and the integrations over p_i give a finite value. But in the region II the possible points of p_i are located on the following surface

$$
\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} - \frac{g'_{tt}}{-\mathcal{D}}\left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}} E\right)^2 = -\left(\frac{E^2}{g'_{tt}} + \mu^2\right), \qquad (17)
$$

which is a hyperboloid, *a non-compact surface*. So $g(E)$ diverges and the integrations over p_i diverges. In case of $g_{tt}^{'} = 0$, the possible points are given by the surface

$$
\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} = \frac{p_\phi - \left(\frac{g_{\phi\phi}E^2}{\mathcal{D}} + \mu^2\right) / \left(\frac{2g_{t\phi}}{\mathcal{D}} E\right)}{\frac{-\mathcal{D}}{2g_{t\phi}E}}, \qquad (18)
$$

which is a elliptic paraboloid and also *non-compact*. Therefore the value of the p_i integrations are divergent. Actually the surface such that ${g'}_{tt} = 0$ is the velocity of the light surface (VLS). Beyond VLS (in region II) the co-moving observer must move more rapidly than the velocity of light. Thus we will assume that the system is in the region I. For example, in case of $\Omega_0 = 0$ the points satisfying ${g'}_{tt} = 0$ are on the stationary limit surface. The region of the outside (inside) of the stationary limit surface corresponds to the region I (II). In case of $\Omega_0 = \Omega_H$, where Ω_H is the angular velocity of the black hole, the region I corresponds to $r_H < r < r_{VLS}$, with $r_{VLS} \sim 1/\Omega_H$ approximately (for Kerr-Newman black hole). VLS is an open, roughly, cylindrical surface [9].

After some calculation we obtain

$$\begin{aligned}
&\beta W \\
&= \int_{region\ I} dr d\theta d\phi \int_{\mu\sqrt{-g'_{tt}}}^{\infty} \frac{dE}{2\pi^2} \frac{\sqrt{g_4}}{(-g'_{tt})^2} E \left[E^2 - (-g'_{tt})\mu^2\right]^{1/2} \ln\left(1 - e^{-\beta(E - q\Gamma)}\right) \\
&= -\beta \int_{region\ I} dr d\theta d\phi \int_{\mu\sqrt{-g'_{tt}}}^{\infty} \frac{dE}{6\pi^2} \frac{\sqrt{g_4}}{(-g'_{tt})^2} \frac{\left[E^2 - (-g'_{tt})\mu^2\right]^{3/2}}{e^{\beta(E-q\Gamma)} - 1}, \qquad (19)
\end{aligned}$$

where we have integrated by parts. In particular when $\Omega_0 = a = e = q = 0$, the expression (19) coincides with the expression obtained by 't Hooft [5] and it is proportional to the volume of the optical space [8]. In case of $q = 0$ it reduces to the result in Ref. [9]. It is easy to see that the integrand diverges as $h \to 0$.

Let $\mu = 0$. For a massless charged scalar field the thermodynamic potential reduces to[13]

$$\begin{aligned}
W &= -\int_{region\ I} dr d\theta d\phi \int_0^{\infty} \frac{dE}{6\pi^2} \frac{\sqrt{g_4}}{(-g'_{tt})^2} \frac{E^3}{e^{\beta(E-q\Gamma)} - 1} \\
&= -\int_{region\ I} dr d\theta d\phi \frac{1}{\pi^2\beta^4} \frac{\sqrt{g_4}}{(-g'_{tt})^2} \sum_{k=1}^{\infty} \left(\frac{e^{kq\beta\Gamma}}{k^4}\right) \\
&= -\int_{regionI} dr d\theta d\phi \frac{\sqrt{g_4}}{\pi^2 \beta_{local}^4} \sum_{k=1}^{\infty} \left(\frac{e^{kq\beta\Gamma}}{k^4}\right), \qquad (20)
\end{aligned}$$

where $\beta_{local} = \sqrt{-g'_{tt}}\beta$ is the reciprocal of the local Tolman temperature [14]. This form is just

the thermodynamic potential of a gas of massless particles with chemical potential $q\Gamma$ at local temperature $1/\beta_{local}$. (We assumed $q\Gamma < 0$. Then for the antiparticle there is superradiance. Thus to obtain a finite value $\mathcal{E} - m\Omega_0 - q\Phi_0$ must great than 0. See Eq. (6).)

Now let us assume that $\Omega_0 = \Omega_H = \frac{a}{r_H^2+a^2}$ and $\Phi_0 = \Phi_H = \frac{er_H}{r_H^2+a^2}$. In this case $\Gamma(r = r_H) = 0$. Thus the leading behavior of the thermodynamic potential for very small h is given by

$$\begin{aligned}
\beta W &\approx -\frac{\pi^2}{90\beta^3}\int d\phi d\theta \int_{r_H+h}^{L} dr\sqrt{g_4}\left\{\left(\frac{g_{\phi\phi}}{-\mathcal{D}}\right)^2 + 2\left(-\frac{g_{t\phi}}{g_{\phi\phi}} - \Omega_H\right)^2 \frac{g_{\phi\phi}^4}{(-\mathcal{D})^3}\right\} \\
&\approx -\frac{\pi^2}{90\beta^3}\frac{A_H}{8\kappa^3}\left\{2\kappa\left(\frac{r_H}{a} + \frac{a}{r_H}\right)\tan^{-1}\left(\frac{a}{r_H}\right)\cdot\frac{1}{h} + \left(\left[-\frac{6\kappa}{r_H} + \frac{2}{r_H^2+a^2} - \frac{3}{r_H^2}\right]\right.\right. \\
&\quad + \left.\left.\left[-\frac{6a\kappa}{r_H^2} - \frac{6\kappa}{a} + \frac{3}{ar_H} - \frac{3a}{r_H^3}\right]\tan^{-1}\left(\frac{a}{r_H}\right)\right)\cdot\ln(h) + \cdots\right\} \\
&\equiv -\frac{\pi^2}{90\beta^3}\frac{A_H}{8\kappa^3}\left\{C\cdot\frac{1}{h} + D\cdot\ln(h) + \cdots\right\},
\end{aligned} \tag{21}$$

where $r_- = M - \sqrt{M^2 - a^2 - e^2}$ or in terms of the proper distance cut-off ϵ

$$\begin{aligned}
\beta W &\approx -\frac{\pi^2}{90\beta^3}\int_{r=r_H} d\phi d\theta\sqrt{g_{\theta\theta}g_{\phi\phi}}\int_{r_H+h}^{L} dr\sqrt{g_{rr}}\left(\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right)^{3/2} \\
&\approx -\frac{\pi^2}{180(\kappa\beta)^3}\frac{A_H}{\epsilon^2}.
\end{aligned} \tag{22}$$

Here A_H is the area of the event horizon, and κ is the surface gravity of the black hole, and ϵ is the proper distance from the horizon to $r_H + h$:

$$A_H = \int_{r=r_H}\sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta d\phi = 4\pi(r_H^2 + a^2), \tag{23}$$

$$\kappa = \frac{1}{2}\frac{r_H - r_-}{(r_H^2 + a^2)} = \frac{(M^2 - a^2 - e^2)^{1/2}}{2M[M + (M^2 - a^2 - e^2)^{1/2}] - e^2}, \tag{24}$$

$$\epsilon = \int_{r_H}^{r_H+h}\sqrt{g_{rr}}dr \approx 2\left(\frac{r_H^2 + a^2\cos^2\theta}{2r_H - 2M}\right)^{1/2}\sqrt{h}. \tag{25}$$

The leading behavior of the entropy S including the contribution of the antiparticle is

$$\begin{aligned}
S &= \beta^2\frac{\partial}{\partial\beta}W \\
&\approx \frac{\pi^2}{90\kappa^3\beta^3}A_H\left\{C\cdot\frac{1}{h} + D\cdot\ln(h)\right\}
\end{aligned} \tag{26}$$

or

$$S \approx 8\frac{\pi^2}{180(\kappa\beta)^3}\frac{A_H}{\epsilon^2}, \tag{27}$$

which diverges as $h \to 0$. The divergences arise from the fact that the density of states for a given E diverges as h goes to zero.

If we take T as the Hartle-Hawking temperature $T_H = \frac{\kappa}{2\pi}$ (In this case the quantum state is the Hartle-Hawking vacuum state $|H\rangle$ [15].) the entropy becomes

$$S \approx 4NA_H \left\{ C \cdot \frac{1}{h} + D \cdot \ln(h) \right\} \tag{28}$$

or

$$S \approx N\frac{A_H}{\epsilon^2}, \tag{29}$$

where N is a constant. The entropy of a complex scalar field diverges quadratically in ϵ as the system approaches the horizon, or it diverges linearly and logarithmically in h. In case of $a = 0 = q$ our result (29) agrees with the result calculated by 't Hooft [5] and with one in Ref. [16]. These facts imply that the leading behavior of entropy (29) is a general form.

As a summary, we showed the leading behavior of the entropy of the complex scalar field in Hartle-Hawking vacuum is proportional to the horizon area, but diverges as the cut-off h goes to zero. The reason of the divergence is that at the horizon the density of states for a given E diverges. The particular points are followings:

1) There is a logarithmic divergence in the coordinate cut-off h.

2) The proper distance cut-off ϵ is dependent on the coordinate θ, which came from the ax-sisymmetric properties of the spacetime.

3) To obtain the entropy proportional to the horizon area, the angular velocity should be equal to Ω_H and the chemical potential should be equal to Φ_H.

4) In case of the extremal rotating black hole ($r_H = r_-$) we can see that the cubic and the quadratic divergences in h appear. However we must consider only the case $a \leq 1/2M$ to obtain a finite value [9].

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